

# Dimension Reduction for the Hyperbolic Space

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## Abstract

A dimension reduction for the hyperbolic space is established. When points are far apart an embedding with bounded distortion into  $H^2$  is achieved.

## 1 Introduction

Dimension reduction algorithms for Euclidean spaces have numerous algorithmic applications. They help to significantly reduce the space required for storing multidimensional data, and thus to improve performance of many algorithms. In this paper, we present a dimension reduction algorithm for the hyperbolic space. Our results show that many existing algorithms for Euclidean spaces that rely on dimension reduction can be also applied to hyperbolic spaces. We refer the reader to a paper of Ailon and Chazelle [1] for background on dimension reduction algorithms and some of their applications. We also refer the reader to a paper of Krauthgamer and Lee [5], which studies combinatorial algorithms for hyperbolic spaces.

For background on hyperbolic geometry see e.g. [3]. For a recent study of convexity and high dimensional hyperbolic spaces see [7].

### 1.1 Our Results

In this paper, we consider the Poincaré half-space model of the hyperbolic space  $H^n$ . Recall that every point is represented as a pair  $(z, x)$ ,  $z \in \mathbb{R}^+$ ,

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$x \in \mathbb{R}^{n-1}$  in this model. The distance between two points  $p_1 = (z_1, x_1)$  and  $p_2 = (z_2, x_2)$  is defined by

$$d(p_1, p_2) = \operatorname{arccosh} \left( 1 + \frac{\|x_1 - x_2\|^2 + (z_1 - z_2)^2}{2z_1 z_2} \right).$$

For brevity, we define  $F(r, z_1, z_2)$  as follows:

$$F_{z_1, z_2}(r) = \operatorname{arccosh} \left( 1 + \frac{r^2 + (z_1 - z_2)^2}{2z_1 z_2} \right).$$

Then

$$d(p_1, p_2) = F_{z_1, z_2}(\|x_1 - x_2\|).$$

Suppose we are given an  $n$ -point subset  $S$  of the hyperbolic space. Let  $T$  be its projection on  $\mathbb{R}^{n-1}$ :

$$T = \{x : (z, x) \in S\}.$$

By the Johnson–Lindenstrauss lemma [4], there exists an embedding of  $T$  into  $O((\log n)/\varepsilon^2)$  dimensional Euclidean space such that for every  $x_1, x_2 \in T$

$$\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| \leq (1 + \varepsilon)\|x_1 - x_2\|.$$

**Theorem 1.1** (Dimension Reduction for  $H^n$ ). *Consider the map  $g : H^n \rightarrow H^{O(\log n)}$  defined by*

$$g(p) \equiv g((z, x)) = (z, f(x)).$$

*Then for every two points  $p_1$  and  $p_2$  at (hyperbolic) distance  $\Delta$ , we have*

$$\Delta \leq d(g(p_1), g(p_2)) \leq \left( 1 + \frac{3\varepsilon}{1 + \Delta} \right) \Delta.$$

**Remark 1.1.** *Since we reduce the hyperbolic case to the Euclidean case, the dimension reduction embedding for  $H^n$  can be computed very efficiently using the Fast Johnson–Lindenstrauss Transform of Ailon and Chazelle [1].*

The following corollary follows from a result of Bonk and Schramm [2].

**Corollary 1.2.** *Let  $X$  be a Gromov hyperbolic geodesic metric space with bounded growth at some scale. Then there exist constants  $\lambda_X$  and  $C_X$  such that every  $n$ -point subset  $S$  of  $X$  roughly quasi-similarly embeds into a  $O((\log n)/\varepsilon^2)$  dimensional hyperbolic space. That is, there exists a map  $\varphi : S \rightarrow H^{O((\log n)/\varepsilon^2)}$  such that for every  $x, y \in S$*

$$\lambda_X d(x, y) - C_X \leq d(\varphi(x), \varphi(y)) \leq (1 + \varepsilon)\lambda_X d(x, y) + C_X.$$

For far apart points we prove the following theorem.

**Theorem 1.3** (Embedding into Hyperbolic Plane). *Let  $S$  be an  $n$  point subset of  $H^n$ . Assume that the distance between every two points in  $S$  is at least  $\frac{\ln(12n)}{\varepsilon}$  then there exists an embedding of  $S$  into the hyperbolic plane  $H^2$  with distortion at most  $1 + \varepsilon$ .*

## 2 Proofs

We start with the proof of the first theorem followed by the proof for the second.

*Proof of Theorem 1.1.* First, since  $F_{z_1, z_2}$  is an increasing function, we have

$$d(g(p_1), g(p_2)) = F_{z_1, z_2}(\|f(x_1) - f(x_2)\|) \geq F_{z_1, z_2}(\|x_1 - x_2\|) = \Delta.$$

On the other hand, by the mean value theorem,

$$d(g(p_1), g(p_2)) \leq F_{z_1, z_2}((1 + \varepsilon)\|x_1 - x_2\|) \quad (1)$$

$$= F_{z_1, z_2}(\|x_1 - x_2\|) + \frac{dF_{z_1, z_2}(\hat{r})}{dr} \cdot \varepsilon\|x_1 - x_2\|, \quad (2)$$

where  $\hat{r} \in (\|x_1 - x_2\|, (1 + \varepsilon)\|x_1 - x_2\|)$ . Let us now bound the derivative of  $F_{z_1, z_2}$ .

$$\begin{aligned} \frac{dF_{z_1, z_2}(\hat{r})}{dr} &= \frac{2\hat{r}}{2z_1z_2} \cdot \frac{1}{\sqrt{t-1}\sqrt{t+1}} \Big|_{t=1+\frac{\hat{r}^2+(z_1-z_2)^2}{2z_1z_2}} \\ &= \frac{2\hat{r}}{\sqrt{\hat{r}^2 + |z_1 - z_2|^2} \sqrt{\hat{r}^2 + |z_1 - z_2|^2 + 4z_1z_2}} \\ &\leq \frac{2}{\sqrt{\hat{r}^2 + |z_1 - z_2|^2 + 4z_1z_2}} \\ &\leq \frac{2}{\sqrt{\|x_1 - x_2\|^2 + |z_1 - z_2|^2 + 4z_1z_2}}. \end{aligned}$$

Here, we used that  $(\operatorname{arccosh} t)' = 1/\sqrt{(t-1)(t+1)}$ . From the identity

$$\frac{\|x_1 - x_2\|^2 + |z_1 - z_2|^2}{2z_1z_2} = \cosh \Delta - 1,$$

and the bound for  $\frac{dF_{z_1, z_2}(\hat{r})}{dr}$  we get an estimate for the additive term in (2)

$$\begin{aligned} \frac{dF_{z_1, z_2}(\hat{r})}{dr} \cdot \varepsilon \|x_1 - x_2\| &\leq \frac{2\|x_1 - x_2\| \varepsilon}{\sqrt{2z_1 z_2 (\cosh \Delta + 1)}} \\ &\leq 2\varepsilon \sqrt{\frac{\|x_1 - x_2\|^2 + |z_1 - z_2|^2}{2z_1 z_2}} \cdot \frac{1}{\sqrt{\cosh \Delta + 1}} \\ &= 2\varepsilon \sqrt{\frac{\cosh \Delta - 1}{\cosh \Delta + 1}} = 2\varepsilon \tanh \frac{\Delta}{2}. \end{aligned}$$

It is easy to see that

$$\tanh t \leq \frac{3t}{1 + 2t}$$

for  $t > 0$ . Therefore, the additive term in (2) is at most

$$\frac{3\varepsilon}{1 + \Delta} \Delta.$$

This concludes the proof.  $\square$

*Proof of Theorem 1.3.* Define  $T = \{x : (z, x) \in S\}$ . By a theorem of Matoušek [6], there exists an embedding  $f : T \rightarrow \mathbb{R}$  of  $\mathbb{R}^{n-1}$  into  $\mathbb{R}$  with distortion at most  $12n$ . We assume that  $f$  is non-contracting and  $\|f\|_{Lip} \leq 12n$ . Consider the embedding  $g : S \rightarrow H^2$  defined by

$$g((z, x)) = (z, f(x)).$$

Clearly,  $g$  is non-contracting. Now we upper bound the Lipschitz norm of  $g$ . Pick two points  $p_1 = (z_1, x_1)$  and  $p_2 = (z_2, x_2)$  at distance  $\Delta$  in  $S$ . Let  $r = \|x_1 - x_2\|$ .

$$\begin{aligned} d(g(p_1), g(p_2)) &= F_{z_1, z_2}(\|f(x_1) - f(x_2)\|) \leq F_{z_1, z_2}(12nr) \\ &\leq \operatorname{arccosh} \left( 1 + 12n \frac{r^2 + |z_1 - z_2|^2}{2z_1 z_2} \right). \end{aligned}$$

Since

$$\frac{r^2 + |z_1 - z_2|^2}{2z_1 z_2} = \cosh \Delta - 1,$$

we have

$$d(g(p_1), g(p_2)) \leq \operatorname{arccosh}(12n \cosh \Delta - (12n - 1)).$$

Observe that

$$\begin{aligned} \cosh t &= \frac{e^t + e^{-t}}{2} \leq \frac{e^t}{2} + \frac{1}{2} \quad (\text{for } t > 0); \\ \operatorname{arccosh} t &= \ln(t + \sqrt{t^2 - 1}) \leq \ln(2t) \quad (\text{for } t > 1). \end{aligned}$$

Therefore,

$$\begin{aligned} d(g(p_1), g(p_2)) &\leq \operatorname{arccosh}(12n \cosh \Delta - (12n - 1)) \leq \ln(2 \cdot 12n \frac{e^\Delta}{2}) \\ &= \Delta + \ln(12n) \leq (1 + \varepsilon)\Delta. \end{aligned}$$

This concludes the proof.  $\square$

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